Theory of controlled quantum dynamics

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Abstract. We introduce a general formalism to obtain localized quantum wavepackets as dynamically controlled systems, in the framework of Nelson stochastic quantization. We show that in general the control is linear, and it amounts to introducing additional time-dependent terms in the potential. In this way one can construct for general systems either coherent packets following classical motion with constant dispersion, or coherent packets following classical motion whose time-dependent dispersion remains bounded for all times. We show that in the operatorial language our scheme amounts to introducing a suitable generalization to arbitrary potentials of the displacement and scaling operators that generate the coherent and squeezed states of the harmonic oscillator.

1. Introduction

The present work addresses the problem of developing a comprehensive theoretical approach to quantum control in the framework of Nelson stochastic quantization, which is currently recognized as an independent and self-consistent formulation of nonrelativistic quantum mechanics in the language of stochastic processes [1–3].

The experimental goal of quantum control is to use radiation to drive matter to a desired target or outcome [4], and several theoretical schemes modelling controlled wavepacket dynamics have been suggested. The unifying theme in all of the current theoretical and experimental schemes is the optimal use of the coherence of laser light to manipulate the quantum mechanical phase relationship among the eigenstates of matter.

From this point of view, the coherent and squeezed states of the harmonic oscillator can be considered as special but paradigmatic examples of controlled wavepacket dynamics [5]. Progress in the femtosecond pulse technology now allows for the realization of quantum control in the laboratory; for example, frequency-chirped femtosecond laser pulses have been synthesized to control the evolution of vibrational wavepackets of the iodine molecule [6].

The potential interest of Nelson stochastic mechanics for the theory of controlled wavepacket dynamics then stems from the fact that the original Nelson quantization scheme, which contained some *ad hoc* assumptions [1], has been later recognized to be a particular instance of classical stochastic control theory. Namely, it has been proven [3] that quantum dynamics can be derived via a stochastic variational principle, by suitably extremizing the classical action along diffusive trajectories replacing the classical deterministic ones. The

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stochastic variational scheme has since been extended and exploited in a number of different contexts [7–9].

For our purposes, what is appealing in the stochastic quantization scheme is the possibility that it offers the possibility to obtain both new results and new insights into old problems, by looking at quantum coherence in terms of general properties of classical diffusion processes.

In particular, variational minimization of the stochastic osmotic uncertainty functional yields the complete structure of the quantum states of minimum uncertainty, i.e. the harmonic-oscillator coherent and squeezed states [10]. The same structure had been derived earlier [11] by simply saturating the osmotic uncertainty inequality [12].

In the present paper we address, in the stochastic quantization approach, the problem of constructing a general framework for controlled wavepacket dynamics, paying special attention to the construction of coherent and squeezed states for general non-harmonic potentials.

To that end, we introduce the conditions of classical motion for the wavepacket centre and the conditions of constant or bounded time-varying dispersion as constraints for the stochastic dynamics in an assigned, generic configurational external potential $V(x)$. We then show that such constraints select a class of Nelson diffusions with classical current velocity and wave-like propagating osmotic velocity [13, 14].

To each Nelson diffusion belonging to such class, there is associated a quantum state that is a solution of the Schrödinger equation in a time-dependent potential $\bar{V}(x, t)$ having a simple relation with the original system $V(x)$. In particular, if we ask for a purely coherent wavepacket solution, i.e. one whose centre $\langle \hat{x} \rangle$ follows exactly the classical motion $x_{cl}(t)$ in $V(x)$ with constant dispersion $\Delta \hat{x}$, then the potential $\bar{V}(x, t)$ is completely determined by the functional form of $V(x)$ and by the knowledge of $x_{cl}(t)$.

Clearly, such 'controlling' potential $\bar{V}(x, t)$ may be, in principle, experimentally fashioned in the laboratory, once a solvable classical potential $V(x)$ is assigned.

Connection with the standard operatorial language is then provided by showing that this new class of controlled coherent states is generated by letting the standard unitary Glauber displacement operator act on any stationary state (for instance the ground state) associated to $V(x)$.

We then extend our scheme to construct coherent wavepackets with bounded time-varying dispersion $\Delta \hat{x}$. By resorting to the stochastic framework we are able to determine the new controlling potential $\tilde{V}(x, t)$ (that in general does not coincide with the controlling potential $\bar{V}(x, t)$ obtained in the case of constant spreading) connected to the classical potential $V(x)$, and to derive the evolution equation for $\Delta \hat{x}$, which turns out to be the classical envelope equation, well known in the theory of charged-particle beam dynamics.

Knowledge of the solution $\Delta x_{cl}(t)$ of the classical envelope equation and of the classical trajectory $x_{cl}(t)$ associated to the potential $V(x)$, determines unambiguously the form of the controlling potential $\tilde{V}(x, t)$.

A suitable unitary operator acting on any stationary state (for instance the ground state) associated to $V(x)$ is then introduced in the coordinate representation in order to construct these controlled states in the standard operatorial language: they are displacement-operator generalized coherent states with time-dependent dispersion.

In fact, this unitary operator acts as the product of two distinct mappings on the ground state of the given potential $V(x)$: the ordinary Glauber displacement operator and a dynamical scaling operator, namely a dynamical squeeze operator. Squeezing is then naturally embedded in this scheme, and the evolution equation for $\Delta \hat{x}$ also yields the dynamical equation controlling the time-evolution of squeezing.

Overall, the above construction provides a natural way to introduce a class of physical coherent and squeezed states associated to general non-harmonic potentials, in the sense of controlled wavepacket dynamics, by imposing suitable requirements on the physical properties of the desired solutions. In this way we provide a specific physical solution to the problem, posed by Schrödinger over 70 years ago, on how to generalize the notion of coherent state beyond harmonic systems.

This paper is organized as follows. In section 2 we give a brief review of the basic ingredients of Nelson stochastic quantization that will be needed in the following sections. In section 3 we describe the structure of the harmonic-oscillator coherent and squeezed states in the stochastic picture. In section 4 we show how to construct general controlled coherent states in the stochastic framework by imposing the constraints of classical motion for the wavepacket centre and of constant wavepacket dispersion, and we derive the explicit expression for the controlling potential $\bar{V}(x, t)$. In section 5 we extend the discussion imposing the constraints of classical motion and of bounded time-varying dispersion and we derive the explicit expression for the displacement and squeeze operators associated to the controlled coherent states with bounded time-varying dispersion. In section 6 we draw our conclusions.

2. Stochastic mechanics

We shall quickly review the basic ingredients of the stochastic formulation of quantum mechanics that will be needed in the following.

This quantization procedure rests on two basic prescriptions; the first one, kinematical, promotes the configuration of a classical system to a conservative diffusion process with diffusion coefficient equal to $\hbar/2m$.

If we denote by $q(t)$ the configurational variable for a point particle with mass m, this prescription reads

$$\mathrm{d}q(t) = v_{(+)}(q(t), t)\,\mathrm{d}t + \sqrt{\frac{\hbar}{2m}}\,\mathrm{d}w(t) \qquad \mathrm{d}t > 0. \tag{1}$$

In the above stochastic differential equation $v_{(+)}$ is a (forward) drift field that is determined by assigning the dynamics, and w is the standard Wiener process.

An intuitive manner to look at equation (1) is to consider it as the appropriate quantum form of the classical kinematical prescription: the Wiener process models the quantum fluctuations that are superimposed on the classical deterministic dynamics.

If we consider the stochastic backward increment of the process $\mathrm{d}q(t) = q(t) - q(t - \mathrm{d}t)$, under very general mathematical conditions the diffusion $q(t)$ admits the backward representation

$$\mathrm{d}q(t) = v_{(-)}(q(t), t)\,\mathrm{d}t + \sqrt{\frac{\hbar}{2m}}\,\mathrm{d}w^*(t) \qquad \mathrm{d}t > 0 \tag{2}$$

where w^* is a time-reversed Wiener process. The probability density $\rho(x, t)$ of the process, defined on the points x of the configuration space, is induced by the conditioned expectations on the increments of the process with respect to the Wiener measure. The backward and forward drifts can then be expressed as stochastic fields on the configuration space through

the conditioned expectations

$$v_{(+)}(x, t) \equiv \lim_{\Delta t \to 0^+} \left\langle \frac{q(t + \Delta t) - q(t)}{\Delta t} \middle| q(t) = x \right\rangle$$

$$v_{(-)}(x, t) \equiv \lim_{\Delta t \to 0^+} \left\langle \frac{q(t) - q(t - \Delta t)}{\Delta t} \middle| q(t) = x \right\rangle. \tag{3}$$

They represent respectively the mean forward (backward) velocity fields.

In the hydrodynamic picture of the process, the drifts are replaced by the osmotic velocity $u(x, t)$,

$$u(x, t) \equiv \frac{v_{(+)}(x, t) - v_{(-)}(x, t)}{2} = \frac{\hbar}{2m} \nabla [\ln \rho(x, t)] \tag{4}$$

and by the current velocity $v(x, t)$,

$$v(x, t) \equiv \frac{v_{(+)}(x, t) + v_{(-)}(x, t)}{2}. \tag{5}$$

Finally, the Fokker–Planck equation for the probability density $\rho(x, t)$ takes the form of the continuity equation

$$\partial_t \rho(x, t) = -\nabla [\rho(x, t) v(x, t)]. \tag{6}$$

The process is completely determined by the the couple $(v(x, t), u(x, t))$ or, equivalently, by the couple $(v(x, t), \rho(x, t))$.

The dynamical prescription is introduced by defining the mean regularized classical action A. In the hydrodynamic Eulerian picture it is a functional of the couple (ρ, v):

$$A[\rho, v] = \int_{t_a}^{t_b} \left[\frac{m}{2} (v^2(x, t) - u^2(x, t)) - V(x, t) \right] \rho(x, t) \, \mathrm{d}^3 x \, \mathrm{d} t \tag{7}$$

where $V(x, t)$ denotes the (possibly time-dependent) configurational external potential.

The equations of motion are then obtained by extremizing A against smooth variations $\delta \rho$, δv vanishing at the boundaries of integration, with the continuity equation taken as a constraint.

After standard calculations one obtains the 'quantum Newton law'

$$\partial_t v(x, t) + (v(x, t) \cdot \nabla) v(x, t) - \frac{\hbar^2}{4m^2} \nabla \left(\frac{\nabla^2 \sqrt{\rho(x, t)}}{\sqrt{\rho(x, t)}} \right) = -\nabla V(x, t) \tag{8}$$

with the current velocity fixed to be a gradient field at all points x where $\rho(x, t) > 0$:

$$v(x, t) = \frac{\nabla S(x, t)}{m} \tag{9}$$

where $S(x, t)$ is a scalar field.

Defining the wavefunction $\Psi(x, t)$ associated to a generic single-particle quantum state in the hydrodynamic form

$$\Psi(x, t) = \sqrt{\rho(x, t)} \exp \left[\frac{\mathrm{i}}{\hbar} S(x, t) \right] \tag{10}$$

it immediately follows that the Schrödinger equation with potential $V(x, t)$ for the complex wavefunction $\Psi(x, t)$ is equivalent to the quantum Newton law together with the continuity equation, i.e. to two real nonlinearly coupled equations for the probability density $\rho(x, t)$

and for the phase $S(x,t)$ (or, alternatively, for the osmotic and current velocities $u(x,t)$ and $v(x,t)$).

Then, to each quantum state there corresponds in stochastic mechanics a diffusion process $q(t)$ with

$$\rho(x,t) = |\Psi(x,t)|^2 \tag{11}$$

and

$$v(x,t) = -\frac{i\hbar}{2m}\nabla\left[\ln\frac{\Psi(x,t)}{\Psi^*(x,t)}\right]. \tag{12}$$

The space integral of equation (8) yields the Hamilton–Jacobi–Madelung equation. It is useful for what follows to write this equation in the form

$$\partial_t S(x,t) + \frac{m}{2}v^2(x,t) - \frac{m}{2}u^2(x,t) - \frac{\hbar}{2}\nabla u(x,t) = -V(x,t). \tag{13}$$

The correspondence between expectations and correlations defined in the stochastic and in the canonic pictures are

$$\begin{aligned}
\langle\hat{x}\rangle &= E(q(t)) & \langle\hat{p}\rangle &= mE(v(x,t)) \\
\Delta\hat{x} &= \Delta q & (\Delta\hat{p})^2 &= m^2[(\Delta u)^2 + (\Delta v)^2].
\end{aligned} \tag{14}$$

In the above relations $\hat{x}$ and $\hat{p}$ are the position and momentum operators in the Schrödinger picture, $\langle\cdot\rangle$ are the expectations of the operators in the given state Ψ, $E(\cdot)$ is the expectation of the stochastic variables in the Nelson state $\{\rho, v\}$ corresponding to the state Ψ, and $\Delta(\cdot)$ are the root mean square deviations.

In the theory of diffusion processes, the functional $(\Delta q)^2(\Delta u)^2$ is known as the osmotic uncertainty product; it shares the remarkable property that it is always greater than or equal to the square of the diffusion coefficient. The following chain inequality then immediately follows:

$$(\Delta\hat{x})^2(\Delta\hat{p})^2 \geqslant m^2(\Delta q)^2(\Delta u)^2 \geqslant \frac{\hbar^2}{4}. \tag{15}$$

The osmotic uncertainty relation and its equivalence with the momentum–position uncertainty were proven in [12].

3. Coherent and squeezed states of the harmonic oscillator

Saturation of the osmotic uncertainty relation (15) yields the Glauber coherent states in the stochastic picture [11]: they are Nelson diffusions of minimum osmotic uncertainty, with constant dispersion Δq. They are characterized by a deterministic classical current velocity:

$$v(x,t) = \langle v(x,t)\rangle = v_{cl}(t) \equiv \dot{x}_{cl}(t) \tag{16}$$

and by an osmotic velocity linear in the process:

$$u(x,t) = -\frac{\hbar}{2m\Delta q}\xi. \tag{17}$$

In equation (16) and from now on we denote the stochastic expectations $E(\cdot)$ with the same symbol $\langle\cdot\rangle$ used for quantum expectations (wherever no confusion arises). We have also introduced the adimensional variable

$$\xi = \frac{x - x_{cl}(t)}{\Delta q} \tag{18}$$

obtained by first shifting the coordinate x by the classical trajectory $x_{cl}(t) = \langle q(t) \rangle$, and then by scaling it through the wavepacket dispersion Δq. In fact, the adimensional configurational variable ξ will play a fundamental role in all of the following.

The probability density $\rho(x, t)$ associated to the state (16), (17) is readily obtained by comparing the expression for $u(x, t)$ given in (17) and the relation (4) linking the probability density with the osmotic velocity. The phase $S(x, t)$ is obtained by comparing the expression for $v(x, t)$ given in (16) and the relation (9) connecting the current velocity with the phase. In this way one reconstructs completely the wavefunction $\Psi(x, t)$. The potential $V(x, t)$, with classical trajectories $x_{cl}(t)$, that solves Schrödinger equation for the state (16), (17) is finally identified by replacing the expressions for $S(x, t)$, $v(x, t)$, and $u(x, t)$ in the Hamilton–Jacobi–Madelung equation (13). This is a typical inverse-problems strategy: given a certain state with some desired features, in this case that of being of minimum osmotic uncertainty with constant dispersion, one looks for the potential that solves the Schrödinger equation and allows for such a state.

We notice that the states of minimum osmotic uncertainty with constant Δq have a purely classical current velocity, so that $\Delta v = 0$. Therefore the minimum osmotic uncertainty is exactly equivalent to the minimum Heisenberg uncertainty. Then, following the procedure outlined above, it is straightforward to recover the complete structure of the Heisenberg minimum uncertainty states, which are the Schrödinger–Glauber coherent states of the harmonic oscillator. We have:

$$\rho(x, t) = \frac{1}{\sqrt{2\pi(\Delta q)^2}} \exp\left[-\frac{(x - x_{cl}(t))^2}{2(\Delta q)^2}\right]$$

$$S(x, t) = mv_{cl}(t)x - \frac{m}{2}v_{cl}(t)x_{cl}(t) - \frac{1}{2}\hbar\omega t$$

$$\Psi(x, t) = \frac{1}{[2\pi(\Delta q)^2]^{\frac{1}{4}}} \exp\left\{-\frac{(x - x_{cl}(t))^2}{4(\Delta q)^2} + \frac{i}{\hbar}\left[mv_{cl}(t)\left(x - \frac{x_{cl}(t)}{2}\right) - \frac{\hbar}{2}\omega t\right]\right\}$$

$$V(x) = \frac{m}{2}\omega^2 x^2 \qquad \omega^2 = \frac{\hbar^2}{4m^2(\Delta q)^4}.$$

(19)

For those readers unfamiliar with the stochastic picture, we recall that in expressions (19) $x_{cl}(t) = \langle \hat{x} \rangle$ and $mv_{cl}(t) = \langle \hat{p} \rangle$.

If we now impose saturation of the osmotic uncertainty, but allow for a time-dependent dispersion Δq, we obtain the harmonic-oscillator squeezed states. They are Nelson diffusions with time-varying Δq, with osmotic velocity $u(x, t)$ still of the form (17), and with current velocity allowing for a term dependent on the time-evolution of the dispersion [11]:

$$v(x, t) = v_{cl}(t) + \xi\frac{\mathrm{d}}{\mathrm{d}t}\Delta q.$$

(20)

The last term in equation (20) is responsible for the quantum anticommutator term appearing in the phase of the squeezed wavepackets, which are quantum states of minimum Schrödinger uncertainty. This can be easily seen as follows. Define the centred position operator $\hat{x}_c = \hat{x} - \langle \hat{x} \rangle$, the centred momentum operator $\hat{p}_c = \hat{p} - \langle \hat{p} \rangle$, and the centred Nelson process $q_c(t) = q(t) - \langle q(t) \rangle$. All these quantities have zero expectation value. Next, consider the quantity $q_c^2(t)$; obviously, $\langle q_c^2(t) \rangle = (\Delta q)^2$. By straightforward calculation one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle q_c^2(t) \rangle = \frac{\mathrm{d}}{\mathrm{d}t}(\Delta q)^2 = 2[\langle q(t)v(x, t) \rangle - \langle q(t) \rangle \langle v(x, t) \rangle] = \frac{\langle \{\hat{x}_c, \hat{p}_c\} \rangle}{m}$$

(21)

where $\{,\}$ denotes the quantum anticommutator, which expresses, when taken between $\hat{x}_c$ and $\hat{p}_c$, the quantum position–momentum correlation. In turn, the latter is the Schrödinger part of the uncertainty, and for the squeezed states it is directly connected with the current uncertainty product in the Nelson stochastic picture. In fact, from equation (20) it is straightforward to see that

$$(\Delta v)^2 = 2 \left(\frac{d}{dt} \Delta q \right)^2. \tag{22}$$

The classical evolution equation for the dispersion Δq can be easily obtained by inserting the expressions (17) and (20) for the stochastic hydrodynamic velocities into the quantum Newton law (8) and the Hamilton–Jacobi–Madelung equation (13); after straightforward manipulations, we obtain

$$\frac{d^2 \Delta q}{dt^2} + m\omega^2 \Delta q = \frac{\hbar^2}{4m^2 (\Delta q)^3} \tag{23}$$

which is the classical equation for the beam envelope, well known from the theory of classical optics and of particle accelerator dynamics.

The expressions (19) for the Gaussian probability density and for the harmonic oscillator potential remain unaltered in the case of minimum uncertainty states with time-dependent dispersion. However, from equations (20) and (21) it follows that the phase of the wavefunction picks up an extra term proportional to the quantum anticommutator:

$$\Psi(x, t) = \frac{1}{[2\pi(\Delta q)^2]^{\frac{1}{4}}} \exp\left\{ -\frac{(x - x_{cl}(t))^2}{4(\Delta q)^2} \right.$$
$$+ i\left[\frac{m}{\hbar} v_{cl}(t)x + m\left(\frac{\langle q(t)v(x,t)\rangle - x_{cl}(t)v_{cl}(t)}{2\hbar} \right)\left[\frac{x - x_{cl}(t)}{\Delta q} \right]^2 \right.$$
$$\left. \left. - \frac{m}{2\hbar} v_{cl}(t)x_{cl}(t) - \frac{1}{2}\omega t \right] \right\}. \tag{24}$$

The above stochastic picture for the harmonic-oscillator coherent and squeezed states can be entirely derived in a stochastic variational approach by extremizing the osmotic uncertainty product against smooth variations of the density $\rho(x,t)$ and of the current velocity $v(x,t)$ [10]. The possibility of extending such variational approach to study local minimum uncertainty behaviours in non-harmonic systems is a current subject of investigation [10].

This concludes our discussion of the coherent and squeezed states of the harmonic oscillator in the framework of stochastic mechanics. We next move to study the problem of how to generalize the concept of coherent and squeezed states to non-harmonic systems, from the same perspective of controlled wavepacket dynamics that we have adopted in this section, with the emphasis focused on the determination of the potential that solves Schrödinger dynamics once a certain state with some desired features has been selected.

4. Controlled coherent quantum wavepackets: constant dispersion

A well known property of the coherent states (16), (17) and of the squeezed states (17)–(20), is that they follow the classical motion in the coherent Glauber sense:

$$\frac{d}{dt}(m\langle v(x,t)\rangle) = -\nabla V(x,t)|_{x=\langle q(t)\rangle}. \tag{25}$$

Again, from a dynamical point of view a coherent state is a wavepacket whose centre follows classical motion not only in the mean, but even along the mean (classical) trajectories, and whose dispersion is either constant, or controlled in its time-evolution (squeezing).

In quantum mechanics the dynamics of mean values obeys the Ehrenfest theorem: as a consequence, the coherent evolution (25) is satisfied exactly if

$$\langle \nabla V(x,t) \rangle = \nabla V(x,t)|_{x=\langle q \rangle}. \tag{26}$$

In the case of quadratic potentials the above constraint is automatically satisfied for any quantum state. For other generic potentials $V(x,t)$, equation (26) in general cannot be satisfied. Our strategy is now to search, in the framework of Nelson stochastic mechanics, for states obeying the coherence constraints of classical motion (25), (26) with constant dispersion Δq.

To this end, we first observe that equation (21) can be recast in the form

$$\Delta q \frac{\mathrm{d}\Delta q}{\mathrm{d}t} = \langle q_c(t) \cdot v(x,t) \rangle \tag{27}$$

where the dot denotes the scalar product between the centred process $q_c(t)$ and the current velocity $v(x,t)$.

Therefore, in stochastic mechanics, we have that a necessary and sufficient condition for a constant dispersion Δq is that the expectation value of the scalar product between the centred configurational process and the current velocity vanishes. We see immediately that a sufficient condition for this to happen is that the current velocity be purely classical, that is, $v(x,t) = v_{cl}(t)$. Other possible choices of current velocities orthogonal to the centred Nelson process could in principle be considered, and they might lead to the definition of new classes of states in quantum mechanics. For the moment, we concentrate on the simplest choice of a classical current velocity.

In stochastic mechanics, due to the existence of the continuity equation, a particular choice of the current velocity selects an entire class of osmotic velocities, and thus of quantum states. In particular, the choice $v = v_{cl}(t)$ that guarantees a constant dispersion, does not restrict the osmotic velocity to be of the minimum uncertainty form (17). Rather, upon insertion of $v_{cl}(t)$ in the continuity equation, one can show that the latter is satisfied by any osmotic velocity (probability density) of the following wave-like propagation form [13]:

$$u(x,t) = \frac{\hbar}{2m\Delta q} G(\xi) \tag{28}$$

where $G(\xi)$ can be any arbitrary adimensional function of the adimensional shifted and scaled configurational coordinate ξ, provided it yields a normalizable probability density $\rho(\xi)$. Keeping in mind relation (4) connecting the density to the osmotic velocity, and recalling that a probability density (being non-negative) can be expressed as the exponential of a real function, we have the general form

$$\rho(\xi) = \frac{\mathcal{N}}{(\Delta q)^d} \exp\left[2R(\xi)\right]. \tag{29}$$

In the above, $\mathcal{N}$ is a positive, adimensional normalization constant, and d denotes the spatial dimension of the system under consideration. As $\nabla_x = \nabla_\xi / \Delta q$, it follows that the adimensional normalizable function $R(\xi)$ is related to $G(\xi)$ by $G(\xi) = 2\nabla_\xi R(\xi)$.

The associated wavefunction is thus of the coherent-state form

$$\Psi_c(x,t) = \frac{\mathcal{N}^{1/2}}{(\Delta q)^{d/2}} \exp\left[R(\xi) + \mathrm{i}\left(\frac{m}{\hbar} v_{cl}(t) \cdot x + \frac{S_0(t)}{\hbar}\right)\right] \tag{30}$$

where we recall that $mv_{cl}(t) = \langle \hat{p} \rangle$ and $S_0(t)$ is an arbitrary time-dependent constant. What are the properties of this class of states selected by the pairs (v, u) of the form (16)–(28)? By construction, they are nonstationary states of constant dispersion. Of course, in general they are not Heisenberg minimum uncertainty states; the latter are recovered only with the choice of the linear form $G(\xi) = -\xi$.

However, it is easily verified that they share with the harmonic-oscillator minimum uncertainty states the remarkable property that they obey the constraint (25) for classical motion except, at most, a constant. To see this in the simplest and most explicit way, let us suppose that $v_{cl}(t)$ is chosen such that its integrals $x_{cl}(t)$ are the classical trajectories of a generic one-dimensional configurational potential $V(x)$. What follows can then be generalized with minor technical complications to higher spatial dimensions.

Suppose then that we are given the stationary states of the quantum mechanical problem associated to $V(x)$. For instance, we consider the ground state $\Psi_0(x, t)$. By a simple scaling argument, it can be cast in the general form

$$\Psi_0(x, t) = \frac{\mathcal{N}_0^{1/2}}{\sqrt{\Delta q_0}} \exp\left[F\left(\frac{x}{\Delta q_0} \right) + \frac{i}{\hbar} E_0 t \right] \tag{31}$$

where $\mathcal{N}_0$ is the adimensional normalization constant, Δq_0 the ground-state dispersion, E_0 the ground-state energy, and F a given adimensional function.

Comparing expressions (30) and (31) we see that the coherent wavefunction $\Psi_c(x, t)$ is obtained from the ground-state wavefunction $\Psi_0(x, t)$ first by identifying the arbitrary function R with F, and then by applying to Ψ_0 the following unitary Glauber-like displacement operator [14]:

$$\hat{D}[x_{cl}(t), v_{cl}(t)] = \exp\left[\frac{i}{\hbar} (S_0(t) - E_0 t) \right] \exp\left(\frac{i}{\hbar} mv_{cl}(t)\hat{x} \right) \exp\left(-\frac{i}{\hbar} x_{cl}(t)\hat{p} \right). \tag{32}$$

This operator, when applied to any wavefunction $\Psi(x, t)$ displaces its space argument x into $x - x_{cl}(t)$ and introduces in the phase the coherent term $mv_{cl}(t)x + S_0(t) - E_0 t$.

We now see that the wavepacket (30) is really a coherent state in the sense that the associated probability density has the same functional form of the ground-state density, so that it shares the same statistics; in particular, the two normalization constants coincide and the constant spreading Δq is just the constant ground-state dispersion Δq_0. Furthermore, the uncertainty product remains constant too and equal to the ground-state uncertainty product, just like the standard harmonic-oscillator coherent states do; in fact the latter are just a particular case of the present construction. Last but not least, the wavepacket centre follows the classical motion $x_{cl}(t)$ in the given configurational potential $V(x)$ according to the Glauber law (25).

The price to be paid for this construction is that these states do not satisfy the time-dependent Schrödinger equation in the originally assigned potential $V(x)$, unless of course the latter is chosen to be a polynomial of degree not greater than two.

However, the controlled states $\Psi_c(x, t)$ of the coherent form (30) are still solutions of the time-dependent Schrödinger equation in a modified potential $\bar{V}(x, t)$ which has a very remarkable relation to the original potential $V(x)$. Namely, taking the wave-like density $\rho(\xi)$, equation (29), and the coherent phase $S_c(x, t) = mv_{cl}(t)x + S_0(t)$ associated to the state (30), and inserting them in the Hamilton–Jacobi–Madelung equation (13) one finds that $\Psi_c(x, t)$ is a solution of the time-dependent Schrödinger equation in the following time-dependent potential:

$$\bar{V}(x, t) = V[x - x_{cl}(t)] + m\ddot{x}_{cl}(t)x. \tag{33}$$

The above expression gives the generic form of a controlling potential that allows for the desired wavepacket solution with coherent and localized dynamics in some previously assigned configurational potential $V(x)$.

Formally, the controlling potential $\bar{V}(x, t)$ is obtained from the original external potential $V(x)$ shifting its argument by the classical trajectories associated to $V(x)$ itself, and then by adding a correcting term linear in x, which is multiplied by the inertia associated to the force field $-\nabla_x V(x)$. The time dependence of the controlling potential is thus parametric via the solutions of the classical equations of motion in the external field $V(x)$.

The question naturally arises about the physical interpretation of this scheme of quantum control. In particular, one may ask what is the meaning of the correcting time-dependent linear term and more generally whether a controlling potential of the form (33) might be realizable in the laboratory.

In principle, the potential (33) is a well-defined object, and its practical realization heavily depends on the actual choice of the original potential $V(x)$: one should then take up a careful case-by-case analysis. However, we can make some general comments on the structure of $\bar{V}(x, t)$ for arbitrary $V(x)$.

We observe that the linear correcting term, superimposed on the original potential, has the simple interpretation of an electric field with amplitude varying in time according to the classical force law in $V(x)$. The problem is then what actual time dependences can be experimentally realized. As for the shift in the argument of $V(x)$, it is in principle feasible, provided again that the assigned time dependence of the coefficients multiplying powers of x can be actually fashioned by some waveform generating set-up; yet, we observe that there might be many situations where the shift could be considered a small perturbation, allowing for Taylor expansion of V or other approximate treatements, for instance in the case of a dynamics taking place in spatial and/or time regions where $x_{cl}(t)$ is a slowly varying quantity.

In conclusion, the states that we have constructed in the stochastic framework are controlled wavepackets with optimized quasi-classical behaviour, which generalize the concept of the harmonic-oscillator coherent states in a precise physical sense. They follow a classical motion with constant dispersion in a given configurational potential $V(x)$ and can be obtained via a unitary transformation of a generic energy eigenstate of $V(x)$. The programming potential $\bar{V}(x, t)$ which must be introduced for the desired states to satisfy quantum dynamics has a simple and intriguing structure, strongly related to the original potential $V(x)$. In particular, its form allows in principle for approximate treatements according to what degree of localization and coherence one desires to accomplish for a generic non-harmonic quantum system.

5. Controlled coherent quantum wavepackets: time-dependent dispersion

We now proceed to consider the case of time-dependent dispersion, that is we consider the more general form of equation (20) for the current velocity of minimum uncertainty, and we again follow the strategy adopted in the previous section for the case of the classical current velocity with constant dispersion.

We expect that the controlled coherent states that we will select by taking the choice (20) for the current velocity should be states following two coupled dynamical equations, equation (25), for the wavepacket center, and an evolution equation for the dispersion Δq, analogous of the envelope equation (23) derived in the harmonic case.

We proceed as follows. We first set the notation by relabelling the dispersion as a

function of time. We put

$$\sigma(t) \doteq \Delta q.$$

Next, inserting equation (20) into the continuity equation (6) we are left with

$$\partial_t \rho(x, t) = \frac{v(\xi, t)\nabla_\xi \rho(x, t) - \dot{\sigma}(t)}{\sigma(t)} \tag{34}$$

whose general solution is only a function of ξ, as can be immediately seen, e.g. by moving into Fourier space. Again, one has selected a class of probability densities of the form (29) as well as a class of osmotic velocities of the wave propagating form (28).

Inserting now the current velocity (20) in the equation of motion (8), by equation (28) we obtain

$$-m\xi\dot{\sigma}(t) + \frac{m}{2}\nabla_x u^2(\xi) + \frac{\hbar}{2}\nabla_x^2 u(\xi) = \nabla_x V(x, t) - \langle \nabla_x V(x, t) \rangle \tag{35}$$

where we exploited the Ehrenfest theorem $\dot{v}_{cl}(t) = -\langle \nabla_x V(x, t)\rangle/m$.

Letting $x = \langle q(t)\rangle = x_{cl}(t)$ (i.e. $\xi = 0$), we are left with

$$\nabla_x V(x, t)|_{x=x_{cl}} - \langle \nabla_x V(x, t)\rangle = \frac{m}{2}\nabla_x u^2(\xi)|_{\xi=0} + \frac{\hbar}{2}\nabla_x^2 u(\xi)|_{\xi=0}. \tag{36}$$

It is straightforward to show that this relation also holds in the case of constant dispersion $v(x, t) = v_{cl}(t)$. The right-hand side is obviously either constant or zero except for singular potentials: in these cases $u(\xi)$ diverges in $\xi = 0$. However, the scheme can also be implemented for singular potentials by taking $\xi = x_{cl}(t)$ rather than $x = x_{cl}(t)$. Explicit examples and applications to both singular and non-singular potentials will be discussed in detail elsewhere.

Given the state (20)–(28) we now want to write explicitly the associated phase $S(x, t)$ and the evolution equation for the dispersion $\sigma(t)$. This is achieved by exploiting the Hamilton–Jacobi–Madelung equation (9): recalling that $v(x, t)$ is the gradient field of $S(x, t)$, equation (20) implies

$$S(x, t) = mv_{cl}(t)x + \frac{m[x - x_{cl}(t)]^2}{2\sigma(t)}\dot{\sigma}(t) + S_0(t). \tag{37}$$

Inserting equations (37), (20), and (28) into equation (13), and taking its expectation, we obtain:

$$\frac{m}{2}\sigma(t)\ddot{\sigma}(t) + \frac{m}{2}\left[\langle v(\xi, t)\rangle^2 - \frac{m}{2}\langle u^2(\xi)\rangle\right] = -\langle V(x, t)\rangle + x_{cl}(t)\langle \nabla_x V(x, t)\rangle - \dot{S}_0(t).$$

We can eliminate in the above expression the time-derivative of the classical action and obtain, after trivial manipulations, an evolution equation for $\sigma(t)$:

$$\ddot{\sigma}(t) - \frac{\langle u(\xi)^2\rangle}{\sigma(t)} = -\frac{\langle \xi\nabla_x V(x, t)\rangle}{m}. \tag{38}$$

By equation (28) for $u(\xi)$ it is immediately seen that

$$\langle u(\xi)^2\rangle = \frac{\hbar^2 K^2}{4m^2\sigma^2(t)} \tag{39}$$

where $K^2 = \int_{-\infty}^{\infty} G^2(\xi)\rho(\xi)\,d\xi$; equation (38) is then the desired equation for the time-evolution of the dispersion, generalizing the classical envelope equation (23). Moreover, it

is coupled, through the gradient of the potential $V(x, t)$, with the equation of motion (25) for the wavepacket centre $\langle q(t) \rangle = x_{cl}(t)$.

The general form of the wavefunction for this class of states is readily obtained by putting together equation (29) for $\rho(\xi)$ and equation (37) for $S(x, t)$:

$$\Psi_s(x, t) = \frac{\mathcal{N}^{1/2}}{[\sigma(t)]^{d/2}} \exp\left[R(\xi) + \frac{i}{\hbar}\left(mv_{cl}(t)x + \frac{m[x - x_{cl}(t)]^2}{2\sigma(t)}\dot{\sigma}(t) + S_0(t) \right) \right]. \qquad (40)$$

We can rewrite this expression in more standard quantum mechanical terms by recalling equation (21), so that

$$\Psi_s(x, t) = \frac{\mathcal{N}^{1/2}}{[\sigma(t)]^{d/2}} \exp\left[R(\xi) + \frac{i}{\hbar}\left(\langle\hat{p}\rangle x + \frac{\langle\{\hat{q}_c, \hat{p}_c\}\rangle}{(2\sigma(t))^2}(x - \langle\hat{q}\rangle)^2 + S_0(t) \right) \right]. \qquad (41)$$

The above wavefunctions $\Psi_s(x, t)$ are realizations of nonstationary states with classical motion and controlled time-dependent spreading. They generalize the harmonic-oscillator squeezed states (24) in the same sense as the states (30) $\Psi_c(x, t)$ generalize the harmonic-oscillator coherent states (19).

We shall now move to the study of the controlling potential that needs to be introduced for the controlled squeezed states $\Psi_s(x, t)$ to satisfy the Schrödinger equation. But first we wish to complete the analogy with the controlled coherent states introduced in the previous section. Namely, we will show that the states (40) can be obtained introducing a proper scaling 'squeeze' operator.

We proceed as follows. We first recall that the harmonic-oscillator squeezed states are generated by the successive application of a scaling squeeze operator and of a Glauber displacement operator to the harmonic-oscillator ground state. We then define the following dynamical scaling operator $\hat{S}[\sigma(t)]$:

$$\hat{S}[\sigma(t)] = \exp\left[i\left(\frac{f(t)}{\hbar}\{\hat{q}, \hat{p}\} \right) \right] \exp\left[i\left(\frac{g(t)}{(\sigma_0)^2}\hat{q}^2 \right) \right] \exp\left[\frac{f(t)g(t)}{2\hbar(\sigma_0)^2}[\{\hat{q}, \hat{p}\}, \hat{q}^2] \right] \qquad (42)$$

where σ_0 denotes the (time-independent) dispersion associated to the ground state $\Psi_0(x, t)$ of a certain assigned configurational potential $V(x)$. Given the $\sigma(t)$ solution of equation (38), the two functions $f(t)$ and $g(t)$ read

$$f(t) = -\frac{1}{2}\ln\left(\frac{\sigma(t)}{\sigma_0} \right) \qquad g(t) = \frac{m}{\hbar}[1 - 2f(t)]^{-1}\frac{d}{dt}\ln\sigma(t). \qquad (43)$$

We see from these relations that the function $f(t)$ plays the role of a dynamical squeezing parameter. We now let $\hat{S}[\sigma(t)]$ act on the ground-state wavefunction $\Psi_0(x, t)$, cast in the form (31), associated to a given configurational potential $V(x)$. We thus define the dynamically scaled wavefunction:

$$\Psi_{sc}(x, t) = \hat{S}[\sigma(t)] \cdot \Psi_0(x, t). \qquad (44)$$

By straightforward algebra, $\{\hat{q}, \hat{p}\} = i\hbar(1 + 2x\,d/dx)$ and $[\{\hat{q}, \hat{p}\}, \hat{q}^2] = -4i\hbar\hat{q}^2$, and one obtains

$$\Psi_{sc}(x, t) = \exp\left[f(t)\left(1 + 2x\frac{d}{dx} \right) \right] \cdot \chi(x, t) \qquad (45)$$

where $\chi(x, t)$ is given by

$$\chi(x, t) = \exp\left[i\frac{g(t)}{(\sigma_0)^2}(1 - 2f(t))x^2 \right] \Psi_0(x, t). \qquad (46)$$

We now exploit the extension to the real axis of the following relation, holding for any analytic function $W(z)$, which was introduced in [20] for the study of q-oscillators coherent states:

$$Q^{x \, d/dx}[W(x)] = W(Qx) \tag{47}$$

with Q a real c-number and W analytic on the real axis. Letting $Q = \exp[2f(t)]$ and $W = \chi$ one is finally left with

$$\Psi_{sc}(x, t) = \exp\left[f(t) + i\frac{g(t)}{(\sigma_0)^2}(1 - 2f(t))e^{4f(t)}x^2\right]\Psi_0(e^{2f(t)}x, t). \tag{48}$$

We then obtain the state $\Psi_s(x, t)$, equation (40), by applying $\hat{D}[x_{cl}(t), v_{cl}(t)]$, equation (30), to $\Psi_{sc}(x, t)$:

$$\Psi_s(x, t) = \hat{D}[x_{cl}(t), v_{cl}(t)](\hat{S}[\sigma(t)] \cdot \Psi_0(x, t)) = \hat{D}[x_{cl}(t), v_{cl}(t)] \cdot \Psi_{sc}(x, t). \tag{49}$$

By recalling equations (43) it is then straightforward to show that $\Psi_s(x, t)$, equation (49), coincides with the wavefunction (40). We have thus proved that the controlled squeezed states (40), (41) can also be introduced by a suitable modification of the displacement-operator approach to the harmonic-oscillator coherent and squeezed states.

The states $\Psi_s(x, t)$ do not satisfy the Schrödinger equation in the assigned potential $V(x)$, although their wavepacket centre does follow the classical motion in the force field generated by $V(x)$.

The situation is completely analogous to that described in the previous section. However, now, due to the presence of a time-varying dispersion, the controlling potential does not coincide with the expression (33). We thus label it $\tilde{V}(x, t)$ to distinguish it from $\bar{V}(x, t)$ introduced in the previous section. After solving the Hamilton–Jacobi–Madelung equation, the new controlling potential reads

$$\tilde{V}(x, t) = V[x - x_{cl}(t)] + m\left(\ddot{x}_{cl}(t) - \frac{\ddot{\sigma}(t)}{\sigma(t)}x_{cl}(t)\right)x + \frac{m\ddot{\sigma}(t)}{2\sigma(t)}x^2. \tag{50}$$

We see that, compared to the controlled coherent case, a controlled squeezed state must be associated to a programming potential with an extra correcting quadratic term; furthermore, the time-dependent coefficients of the correcting terms acquire a more complicated structure as they now depend not only on the classical trajectories $x_{cl}(t)$ but also on the solutions $\sigma(t)$ of the generalized classical envelope equation (38).

The physical interpretation of the quadratic correcting term can be given simply in terms of a diamagnetic interaction superimposed by the controller in the laboratory to the previously existing external interaction $V(x)$. About the experimental feasibility of such a controlling set-up we can repeat in principle what we have observed in the previous section for the linear controlling potential. We will further comment on both the linear and the quadratic controlling potentials in our conclusions.

6. Conclusions and outlook

At this point let us summarize our results.

Working in the framework of Nelson stochastic quantization, we have introduced a class of controlled coherent states (constant dispersion) and a class of controlled squeezed states (bounded time-evolution of the dispersion).

The wavepacket centres follow a classical evolution in a generic preassigned time-independent external potential V, while the states obey Schrödinger dynamics in a time-dependent potential that has a simple relation to V. For constant dispersion such a time-dependent potential (which we have named a controlling or programming potential) presents a linear correcting term to the original time-independent potential, see equation (33). For the bounded time-dependent dispersion, the controlling potential presents a linear plus a quadratic correcting term, see equation (50).

For the controlled squeezed states, the evolution equation controlling the spreading of the wavepacket is the classical envelope equation of classical optics, and it is naturally coupled with the classical evolution equation for the wavepacket centre. The solutions of the two classical equations enter as (time-dependent) parameters in the programming potentials.

We have also shown that the controlled states can be obtained through a particular extension of the displacement-operator coherent and squeezed states of the harmonic oscillator by defining a suitable dynamical scaling (squeeze) operator.

The consequent dilatations or contractions of the wavepacket width are then shown to be controlled by a single adimensional squeezing parameter $f(t)$.

A comment is due at this point. It is well known that generalized coherent states can be obtained extending the three different existing approaches to the definition of the harmonic-oscillator coherent states: they are, respectively, the minimum-uncertainty, the annihilation-operator, and the displacement-operator method.

The states obtained by extension of these three methods are in general different [15].

The displacement-operator coherent states are those preserving most of the properties of the harmonic-oscillator coherent states: they are still overcomplete and still enjoy a resolution of unity. Moreover, we have shown in the present paper that by a proper choice of the Glauber parameter α entering the Glauber displacement operator, they follow a classical motion without dispersion.

As to squeezed states, an extension of the minimum-uncertainty and annihilation-operator methods to arbitrary nonlinear systems was carried out by Nieto and collaborators [16]. They also introduced an extension of the minimum-uncertainty method to obtain generalized coherent states [17].

However, an extension of the displacement-operator method to obtain generalized squeezed states runs into difficulties [18] and is still missing, although some progress in that direction has recently been made by Nieto and Truax for systems allowing for Holstein–Primakoff or Bogoliubov transformations [19].

Here we have introduced a possible extension of the displacement-operator method by constructing squeezed states via a structure of controlling potentials. In this sense, we have given a particular answer to the problem posed by Schrödinger exactly 70 years ago, of whether it is possible to construct coherent-states solutions for general non-quadratic potentials. We showed that it is possible, at the price of modifying the original potential in a very definite way: through the external action of a correcting interaction, the original system can follow exactly and indefinitely a coherent (or a squeezed) dynamics.

In this paper we outlined the general features of the method: elsewhere [21] we will present the construction of coherent and squeezed states in the sense of Schrödinger for a wide sample of potentials of physical and conceptual interest.

It might be worth noting that we have chosen for simplicity the ground state to generate controlled coherent and squeezed states by the action of operators (32) and (42); it is, however, immediately seen that their application on any stationary state yields again controlled coherent and squeezed states of the form (30) and (40).

We remark that equation (39) represents the 'stochastic squeezing' condition satisfied by our states. Namely, it expresses the complementary time dependence of the spreading $\sigma(t)$ and of the osmotic velocity uncertainty Δu.

It is easily seen that in the canonic picture equations (14), (20), and (39) imply $\Delta\hat{q}^2\Delta\hat{p}^2 = K^2 + L^2(t)$, with $L(t) = m\Delta\hat{q}\,\mathrm{d}(\Delta\hat{q})/\mathrm{d}t$.

The reciprocal variation in time of $\Delta\hat{q}$ and $\Delta\hat{p}$ is then ruled by $\Delta\hat{q}$ itself, determined as the solution of equation (22) with the initial condition $\Delta\hat{q}_0$. In this way squeezing is introduced as a self-consistent prescription on the dynamical evolution of the wavepacket spreading.

In conclusion, our scheme of coherent and squeezed states via programming potentials provides an instance of the so-called theory of quantum control (or 'controlled quantum mechanics'), in the sense that, given a desired quantum solution (e.g. a squeezed state), one can provide a theoretical framework to describe what dynamical system (potential) must be introduced to produce such a state.

This theoretical model is deeply rooted in the ideas and techniques of the theory of stochastic optimal control: a probabilistic approach to the description and the construction of quasi-deterministic structures, as generalized coherent and squeezed states, must involve an optimization procedure (minimization of the noise) and an external dynamical monitoring (programming potentials) of the non-deterministic system under study (holding the wavepacket localized along the classical trajectory).

We are currently studying the possibility of applying our scheme to two very real and important problems in the field of quantum control: Rydberg wavepackets [22, 5] and molecular pseudo-Gaussian states [6]. In both instances, the experimental situation consists of a laboratory-fashioned interaction (femtosecond laser pulses) superimposed on existing natural interactions (the atomic and molecular potentials) which gives rise to an approximately coherent and localized dynamics of the wavepackets.

Our scheme of quantum control seems potentially capable of suggesting from a general theoretical framework the optimal interactions that should be fashioned to obtain the best control on the dynamics (the highest degree of coherence and localization). Namely, it would be interesting to verify the actual experimental feasibility of the programming potentials (33)–(50) and of the associated controlled coherent and squeezed states for some specific physical systems such as the ones mentioned above. We will report elsewhere [23] about the work currently in progress on these subjects.

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